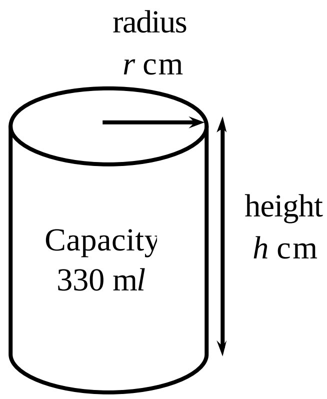
Max & Min Problems

Some problems involving maximising or minimising can be solved from graphs using a graphic calculator or a spreadsheet. Here is one example:

Example A drinks manufacturer plans to sell drinks in a cylindrical can. Each can is to have a capacity of 330 ml.
The manufacturer wants to minimise the material used to make the can.
Find the radius and height of the can that would have the least surface area.



The volume and surface area formulae for a cylinder are:

Volume $V = \pi r^2 h$ Surface Area $S = 2\pi r^2 + 2\pi rh$

Since the volume = 330 ml = 330 cm^3 $330 = \pi r^2 h$ which rearranges to give $h = \frac{330}{\pi r^2}$

Substituting for h in the formula for the surface area gives a formula for S in terms of just one variable, r:

$$S = 2\pi r^2 + 2\pi r \times \frac{330}{\pi r^2} \quad \text{which simplifies to} \quad S = 2\pi r^2 + \frac{660}{r}$$

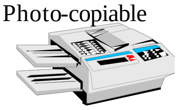
Excel or a graphic calculator can be used to draw a graph of S against r.

The Excel formula that gives values of r at intervals of 0.1 cm and the corresponding values of S are shown below. Use this formula or enter the formula $y = 2\pi x^2 + \frac{660}{x}$ into your graphic calculator to draw a graph of S against r for values of r from 0.1 cm to 10 cm.

	A	B
1	***r***	***S***
2	0.1	=2*PI()*A2^2+660/A2
3	=A2+0.1	=2*PI()*A3^2+660/A3
4	=A3+0.1	=2*PI()*A4^2+660/A4
5	=A4+0.1	=2*PI()*A5^2+660/A5
6	=A5+0.1	=2*PI()*A6^2+660/A6
7	=A6+0.1	=2*PI()*A7^2+660/A7
8	=A7+0.1	=2*PI()*A8^2+660/A8
9	=A8+0.1	=2*PI()*A9^2+660/A9
10	=A9+0.1	=2*PI()*A10^2+660/A10

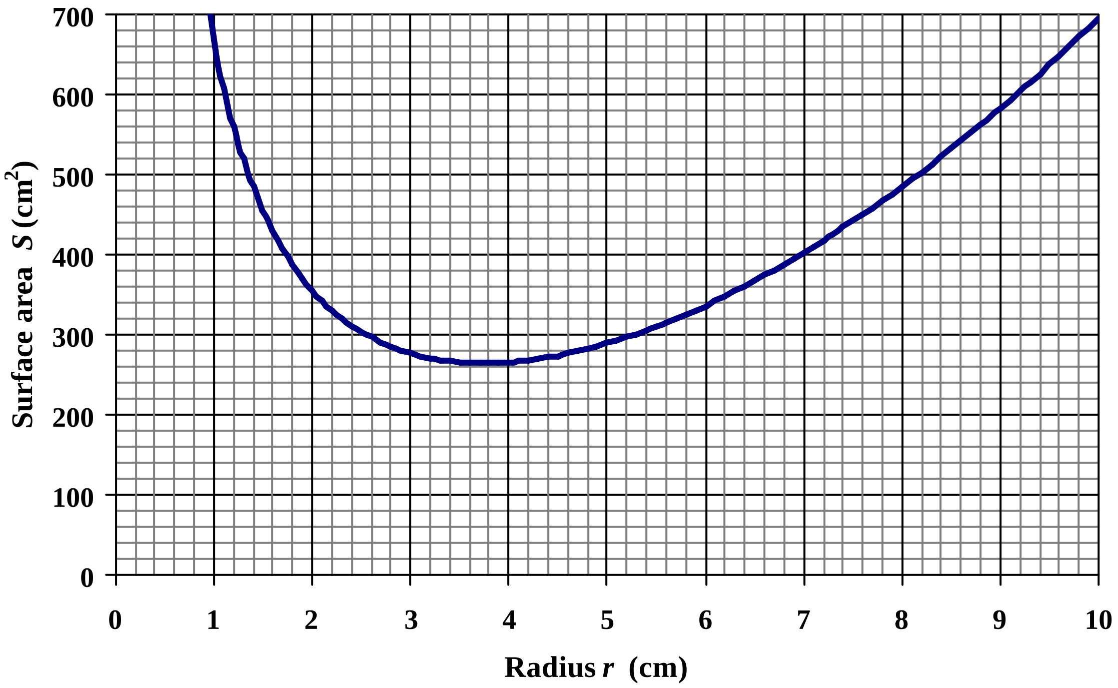
Note that you only need to enter 0.1 into A2 and the formulae shown into B2 and A3, then use 'fill down' to complete the rest of the table down to r = 10.

The graph you should get is shown on the following page.

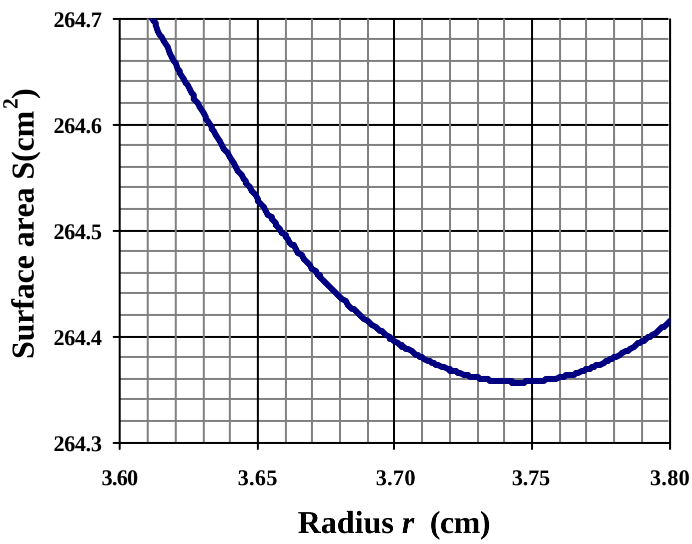


The graph shows that the minimum surface area is about 260 cm^2 and this will occur when the radius is approximately 3.7 cm. You can also see this from the table of values in Excel.
The same result can be obtained from a graphic calculator by using the Zoom and Trace functions near the minimum point on the graph.

Surface area of a can



Surface area of a can



More accurate values can be obtained by using smaller increments in the value of r in Excel or by zooming in further on your graphic calculator.

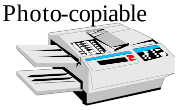
Try this to obtain the values:
r = 3.745 and S = 264.4

The can's height can be found from $h = \frac{330}{pr^2}$.

Substituting r = 3.745 gives h = 7.490

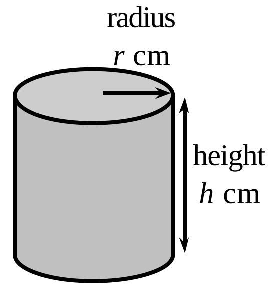
The can with the minimum surface area has radius 3.74 cm and height 7.49 cm (to 3 sf).

(Check that this does give a volume of 330 cm^3 as required.)



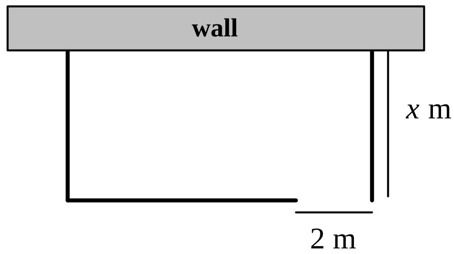
Use a graphic calculator or Excel to solve these problems:

1 (a) A soft drinks manufacturer wants to design a cylindrical can to hold half a litre (500 cm^3) of drink.
Find the minimum area of material that can be used to make the can and the corresponding dimensions of the can.



(b) Repeat part (a) for a can to hold 1 litre of drink.

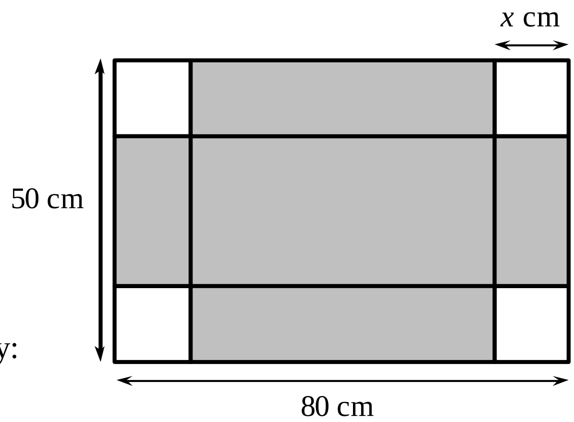
2 A farmer has 100 metres of fencing to use to make a rectangular enclosure for sheep as shown.
He will use an existing wall for one side of the enclosure and leave an opening of 2 metres for a gate.



a) Show that the area of the enclosure is given by:
$A = 102x - 2x^2$

b) Find the maximum possible area and the value of x that gives this area.

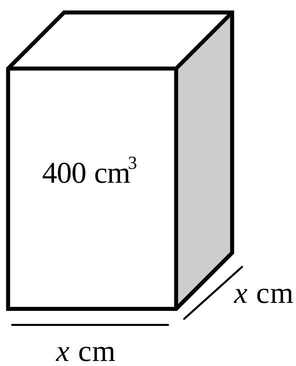
3 An open-topped box is to be made by removing squares from each corner of a rectangular piece of card and then folding up the sides.



a) Show that if the original rectangle of card measured 80 cm by 50 cm and the squares removed from the corners have sides x cm long, then the volume of the box is given by:
$V = 4x^3 - 260x^2 + 4000x$

b) Find the maximum possible volume and the corresponding value of x.

4 (a) A **closed** tank is to have a square base and capacity 400 cm^3



(i) Show that the total surface area of the container is given by:

$$S = 2x^2 + \frac{1600}{x}$$

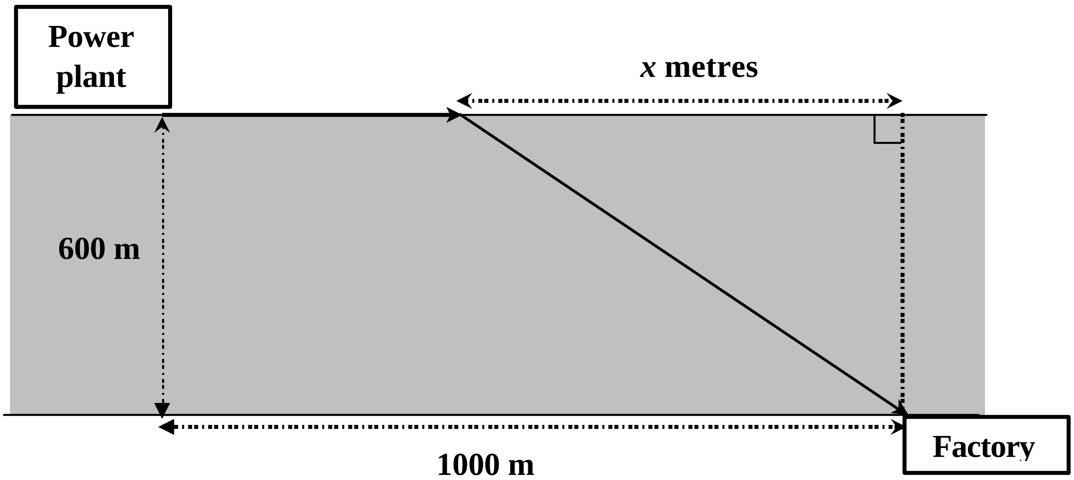
(ii) Find the minimum surface area and the value of x that will give this surface area.

(b) Find the minimum surface area of an **open-topped** tank with a square base and capacity 400 cm^3 and the dimensions of the tank with this surface area.

5 An underground power line is to run from a power plant at one side of a river to a factory at the other side, 1000 metres downstream. The river is 600 m wide and has straight banks.

The sketch below shows the proposed route of the power line. It follows the river bank for some distance before crossing the river to the factory.

The cost of running the line under land is £40 per metre and the cost under water is £50 per metre. It is required to find the route that will cost the least.



a) Find the total cost of the line in terms of x.

b) Find the route that gives the minimum cost.

c) What is the minimum cost?

Teacher Notes

Unit Advanced Level, ***Working with algebraic and graphical techniques***

Notes

The problems included in this resource are intended to be solved using a spreadsheet or graphic calculator. The Powerpoint presentation with the same name shows the method for the example on pages 1 and 2 and can be used as an introduction to this type of work. You could also demonstrate how the formulae are entered and the graph is drawn on a spreadsheet. Alternatively students could work through the example themselves using a spreadsheet or graphic calculator before trying the other problems.

Some of these problems and other problems are used in the resource called **'Maxima and Minima'** which is on the ***skill activities*** page in the ***Modelling with calculus*** section of the website. Students who are also studying calculus could compare the graphical method with the use of differentiation.

Answers (to 3sf)

1 a) Minimum surface area = 349 cm^2 when the radius is 4.30 cm and the height is 8.60 cm
b) Minimum surface area = 554 cm^2 when the radius is 5.42 cm and the height is 10.8 cm
2 b) Maximum possible area = 1300 m^2 when x = 25.5 (m)
3 b) Maximum possible volume = 18 000 cm^3 when x = 10 (cm)
4 a) (ii) Minimum surface area = 326 cm^2 when x = 7.37 (cm)
b) (ii) Minimum surface area = 259 cm^2 when x = 9.28 (cm)
5 a) $C = 40\,000 - 40x + 50\sqrt{x^2 + 360\,000}$
b) Minimum cost occurs when x = 800 (m)
c) Minimum cost = £58 000