APR in more difficult cases

You should already know how to calculate the APR in the simplest case where a sum of money is borrowed at a particular time and paid back, with interest, in a single payment at a later date using the following formula

$$\boldsymbol{C = \frac{A}{(1+i)^n}}$$

which gives the APR, i, as a decimal when a loan of £C is paid back by a single repayment £A after n years.

In more difficult cases when the repayment will be made in more than one instalment, the formula involves a summation. For example, when there are 4 annual instalments, A_1, A_2,, A_3, A_4, and the formula becomes

$$C = \frac{A_1}{1+i} + \frac{A_2}{(1+i)^2} + \frac{A_3}{(1+i)^3} + \frac{A_4}{(1+i)^4}$$

General APR formula for cases involving more than one instalment

$$C = \sum_{k=1}^{m} \left(\frac{A_k}{(1+i)^{(t_k)}} \right)$$

where i is the APR expressed as a decimal, k is the number identifying a particular instalment, A_k is the amount of the instalment k, t_k is the interval in years between the payment of the instalment and the start of the loan.

Checking a given APR value

This simply requires substitution of the given value of the APR into the formula to check whether it gives the correct value of the loan, C.

Example

A loan of £5000 is repaid in three equal annual instalments of £2000.
The APR is quoted as 9.7%. Is this correct?

For 3 annual instalments the formula is $C = \frac{A_1}{1+i} + \frac{A_2}{(1+i)^2} + \frac{A_3}{(1+i)^3}$

Substituting $i = 0.097$ and $A_1 = A_2 = A_3 = 2000$ gives:

$$C = \frac{2000}{1.097} + \frac{2000}{1.097^2} + \frac{2000}{1.097^3} = £1823.15 + £1661.95 + £1514.99$$

$$= £5000.09$$

Note these are the present values of the 3 future instalments.

As this is very near £5000, the statement that the APR is 9.7% is likely to be correct.

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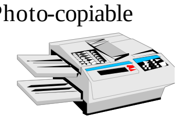

If you wish to be **certain** that 9.7% is correct to 1 decimal place, then you can do this by finding the value of C when $i = 0.0965$ and when 0.0975 as shown below:

When $i = 0.0965$ $$C = \frac{2000}{1.0965} + \frac{2000}{1.0965^2} + \frac{2000}{1.0965^3} = 5005$$

When $i = 0.0975$ $$C = \frac{2000}{1.0975} + \frac{2000}{1.0975^2} + \frac{2000}{1.0975^3} = 4996$$

Can you use your calculator in an efficient way to check these values?

As the first of these gives a loan value above £5000 and the second gives a loan value below £5000, the value of i must lie between 0.0965 and 0.0975. So the APR must lie between 9.65% and 9.75% i.e. the APR is 9.7% correct to 1 decimal place.

Some to try:

1 Check the value of the APR given in each of the following advertisements:

Borrow £10 000.
Repay in 2 annual instalments of £6000 and £7200
APR 20%

Borrow £3600.
Repay in 2 annual instalments of £2250 and £2025
APR 12.5%

2 A borrower repays a debt of £4000 in 3 equal annual instalments of £1500.
Show that the APR is 6.1% correct to 1 decimal place.

3 A loan of £12 000 is repaid in 4 equal annual instalments of £4000.
Show that the APR is 12.6% correct to 1 decimal place.

4 A debt of £11 000 is repaid in 4 annual instalments of £2000, £3000, £4000 and £5000. Show that the APR is 9.0% correct to 1 decimal place.

Finding a value for the APR

When there is only one repayment, the formula for C can be rearranged to find a value for i relatively easily – you should already have used this idea previously. However, when there is more than one instalment, it is not possible to do this and finding a value for the APR becomes much more difficult. One method that can be used is called the interval bisection method. Essentially this involves starting with a range of possible values for i and then repeatedly halving this range until the range becomes so small that it is possible to give an accurate value for i and hence the APR. If you require a very accurate value for i the method is time-consuming and tedious to carry out by hand and the more instalments there are, the worse it gets. The use of a computer makes this much less onerous. The next example explains how the method works using the relatively simple case where a loan is repaid by 4 instalments. Note that this is the most difficult case that the specification suggests you will need, but in real life things can become far more complex.

Finding the APR using the interval bisection method

This 'trial and improvement' method is summarised below:

- Choose an interval of values within which you believe the APR lies.
- Substitute the *i* value at each end of your interval into $\boldsymbol{C} = \sum_{k=1}^{m} \left(\frac{A_k}{(1+i)^{\left(t_k\right)}} \right)$

  to check that the range you have chosen does include the value of *i*.
- Next use the mid-point of your range as *i* and find *C* using $\boldsymbol{C} = \sum_{k=1}^{m} \left(\frac{A_k}{(1+i)^{\left(t_k\right)}} \right)$
- If the calculated value of ***C*** is **too low**, this implies that the value used for ***i*** was **too high**, so you now know that the correct value of ***i* lies in the lower half of the interval you used**.
  If the calculated value of ***C*** is **too high**, this implies that the value used for ***i*** was **too low**, so you now know that the correct value of ***i* lies in the upper half of the interval you used**.
- Repeat the last two steps using the new interval within which you know the correct value of *i* lies. The fact that this is half of the previous interval gives this method its name of '**interval bisection**'.
- Repeat the process again and again, until the interval is narrow enough to give you an accurate value of *i*. The number of steps this takes will depend on how accurate you want your value for the APR to be.

Example
A loan of £24 000 is repaid in 4 annual instalments of £7500, £7500, £8000 and £8000. Use interval bisection to find the APR correct to 1 decimal place.

Solution
Suppose we start by trying the interval 6% to 12%.

When $i = 0.06$, $$C = \frac{7500}{1.06} + \frac{7500}{1.06^2} + \frac{8000}{1.06^3} + \frac{8000}{1.06^4} = 26\,804$$

When $i = 0.12$, $$C = \frac{7500}{1.12} + \frac{7500}{1.12^2} + \frac{8000}{1.12^3} + \frac{8000}{1.12^4} = 23\,454$$

As the true value of *C*, £24 000 lies between these values, then the true value of *i* must lie between 0.06 and 0.12.

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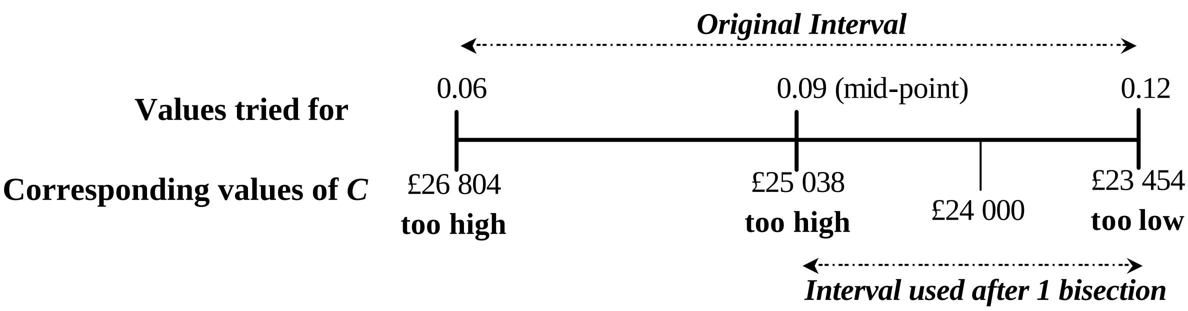
1st bisection

The mid-point of the interval $0.06 < i < 0.12$ is 0.09

When $i = 0.09$, $$C = \frac{7500}{1.09} + \frac{7500}{1.09^2} + \frac{8000}{1.09^3} + \frac{8000}{1.09^4} = 25\,038$$

This is too high, indicating that 0.09 is too low, so i must lie between 0.09 and 0.12.

This sketch illustrates the results so far:



2nd bisection

The mid-point of $0.09 < i < 0.12$ is 0.105

When $i = 0.105$, $$C = \frac{7500}{1.105} + \frac{7500}{1.105^2} + \frac{8000}{1.105^3} + \frac{8000}{1.105^4} = 24\,225$$

This is still too high, so 0.105 is too low and i must lie between 0.105 and 0.12.

3rd bisection

The mid-point of $0.105 < i < 0.12$ is 0.1125

When $i = 0.1125$, $$C = \frac{7500}{1.1125} + \frac{7500}{1.1125^2} + \frac{8000}{1.1125^3} + \frac{8000}{1.1125^4} = 23\,834$$

This is too low, so 0.1125 is too high and i must lie between 0.105 and 0.1125.

This means that the APR must lie between 10.5% and 11.25%.
Hence the **APR = 11% to the nearest %.**
For a more accurate value of the APR, the process can be continued:

4th bisection

The mid-point of $0.105 < i < 0.1125$ is 0.10875

When $i = 0.10875$, $$C = \frac{7500}{1.10875} + \frac{7500}{1.10875^2} + \frac{8000}{1.10875^3} + \frac{8000}{1.10875^4} = 24\,028$$

This is too high, so 0.10875 is too low, so i must lie between 0.10875 and 0.1125.

5th bisection
The mid-point of $0.10875 < i < 0.1125$ is 0.110625

When i = 0.110625,

$$C = \frac{7500}{1.110625} + \frac{7500}{1.110625^2} + \frac{8000}{1.110625^3} + \frac{8000}{1.110625^4} = 23\,931$$

This is too low, so 0.110625 is too high and i must lie between 0.10875 and 0.110625.

6th bisection
The mid-point of $0.10875 < i < 0.110625$ is 0.1096875

When i = 0.1096875,

$$C = \frac{7500}{1.1096875} + \frac{7500}{1.1096875^2} + \frac{8000}{1.1096875^3} + \frac{8000}{1.1096875^4} = 23\,980$$

This is too low, so 0.1096875 is too high and i must lie between 0.10875 and 0.1096875.

7th bisection
The mid-point of $0.10875 < i < 0.1096875$ is 0.10921875

When i = 0.10921875,

$$C = \frac{7500}{1.10921875} + \frac{7500}{1.10921875^2} + \frac{8000}{1.10921875^3} + \frac{8000}{1.10921875^4} = 24\,004$$

This is slightly too high, so 0.10921875 is too low, so i must lie between 0.10921875 and 0.1096875.

8th bisection
The mid-point of the range $0.10921875 < i < 0.1096875$ is 0.109453125

When i = 0.109453125,

$$C = \frac{7500}{1.109453125} + \frac{7500}{1.109453125^2} + \frac{8000}{1.109453125^3} + \frac{8000}{1.109453125^4} = 23992$$

This is slightly too low, so 0.109453125 is too high, so i must lie between 0.10921875 and 0.109453125.

This means that the APR must lie between 10.92% and 10.945%.
Hence the **APR = 10.9% to 1 decimal place.**

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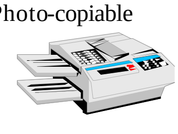

Some to try:

1 A borrower repays a debt of £4000 in 2 annual instalments of £2300.
  (a) Show that the APR lies between 8% and 12%
  (b) Use the interval bisection method to find the APR to the nearest %.

2 A loan of £8000 is repaid in 3 equal annual instalments of £3000.
  (a) Show that the APR lies between 8% and 10%.
  (b) Use the interval bisection method to find the APR correct to 1 decimal place.

3 A debt of £12 500 will be repaid in 4 annual instalments of £4000, £4250, £4500 and £4750.
  (a) Show that the APR lies between 10% and 20%.
  (b) Use the interval bisection method to find the APR correct to 1 decimal place.

4 A loan of £6000 is repaid in 3 annual instalments of £2000, £2500 and £3000.
  (a) Use the interval bisection method to find the APR to the nearest %.
  (b) Continue to use the interval bisection method until you find the value of the APR correct to 1 decimal place.

5 A debt of £3000 is repaid in 4 equal annual instalments of £1000.
  (a) Use the interval bisection method to find the APR to the nearest %.
  (b) Continue to use the interval bisection method until you find the value of the APR correct to 1 decimal place.

6 A borrower repays a loan of £7500 by paying £5000 after one year and £4000 after another year. Find the APR correct to 1 decimal place.

7 To repay a loan of £5500 a borrower pays annual instalments of £1000, £1500, £2000 and £2500. Find the APR correct to 1 decimal place.

8 A lender offers a choice of two ways of repaying a loan of £10 000.

Repayment Method A: Repay in 3 annual instalments of £4000, £4500 and £5000
Repayment Method B: Repay in 4 equal annual instalments of £3500.

(a) Find the total amounts repaid by each method.
(b) Calculate the APR in each case correct to 1 decimal place.
(c) (i) Give a reason why a borrower may prefer to use Method A to repay the loan of £10 000.
    (ii) Give a reason why a borrower may prefer to use Method B to repay the loan of £10 000.

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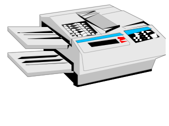

Teacher Notes

Unit Advanced Level, Mathematical Principles for Personal Finance

Notes on Activity

An earlier Nuffield resource for this FSMQ called '**APR – Annual Percentage Rate**' shows learners how to find the APR of a loan in the simplest case where the loan is repaid in a single instalment. This resource involves checking and finding the APR in more difficult cases when a loan or debt is repaid in 2, 3 or 4 annual instalments.

Pages 1 and 2 show how checks can be made by substituting given values of *i* into an equation of the form:

$$C = \frac{A_1}{1+i} + \frac{A_2}{(1+i)^2} + \frac{A_3}{(1+i)^3} + \frac{A_4}{(1+i)^4}$$

Pages 3 to 5 then show how the APR can be found using the interval bisection method to solve the above equation for *i*.

There are also examples for students to try on pages 2 and 6 – the answers to those on page 6 are given below.

The accompanying spreadsheet gives two worksheets that show how spreadsheet formulae can be used to find the APR in the example that starts on page 3. This can be used to demonstrate to learners how a spreadsheet can be used to find an APR or learners can use it themselves to check the answers to the questions on page 6.

Examples of APRs in more complex cases can be found in **'Credit Charges and APR'** which is available from the Office of Fair Trading (www.oft.gov.uk).
If you have time you could ask students to check values of APR that are given in real advertisements for credit cards and mortgages as well as loans.

Answers **Page 6**

1 (b) 10%
2 (b) 6.1%
3 (b) 14.5%
4 (a) 11% (b) 11.2%
5 (a) 13% (b) 12.6%
6 13.6%
7 9.0%
8 (a) Method A £13 500 Method B £14 000
(b) Method A 16.0% Method B 15.0%
(c) (i) A borrower may prefer Method A because it keeps repayments costs to a minimum.
(ii) A borrower may prefer Method B because it spreads the repayments over a longer time and the size of each instalment is lower.

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